Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 201 (2007) 300–313
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Mechanical quadrature methods and their extrapolation for solving first kind Abel integral equations ☆

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Received 26 April 2005; received in revised form 16 February 2006

Abstract

This paper presents high accuracy mechanical quadrature methods for solving first kind Abel integral equations. To avoid the ill-posedness of problem, the first kind Abel integral equation is transformed to the second kind Volterra integral equation with a continuous kernel and a smooth right-hand side term expressed by weakly singular integrals. By using periodization method and modified trapezoidal integration rule, not only high accuracy approximation of the kernel and the right-hand side term can be easily computed, but also two quadrature algorithms for solving first kind Abel integral equations are proposed, which have the high accuracy $O(h^2)$ and asymptotic expansion of the errors. Then by means of Richardson extrapolation, an approximation with higher accuracy order $O(h^3)$ is obtained. Moreover, an a posteriori error estimate for the algorithms is derived. Some numerical results show the efficiency of our methods.

Keywords: The first kind Abel integral equation; Quadrature method; Asymptotic expansion; Extrapolation

1. Introduction

The first kind Abel integral equation

$$\int_0^x \frac{H(x,y)}{(x-y)^\alpha} f(y)\,\mathrm{d}y = g(x) \quad (0 \leqslant x \leqslant 1,\ 0<\alpha<1) \tag{1.1}$$

frequently appears in many physical and engineering problems, e.g., semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics, etc. (see [12]). If $H(x,y) = 1/\Gamma(1-\alpha)$, (1.1) can be written in the form

$$J^{1-\alpha} f = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{1}{(x-y)^\alpha} f(y)\,\mathrm{d}y = g(x), \tag{1.2}$$

☆ Supported by the National Natural Science Foundation of China (10171073).
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0377-0427/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.cam.2006.02.021

which is called the fractional integral equation of order $1-\alpha$. There is a vast literature on fractional calculus and their applications, e.g., Gorenflo [9,10] and Gorenflo and Mainardi [11]. By means of fractional derivative, the solution of (1.2) can be expressed as

$$f(x)=D^{1-\alpha}g(x)=\frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{g(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{g'(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y+\frac{g(0^+)}{\Gamma(\alpha)}x^{\alpha-1}, \tag{1.3}$$

which implies that $f(x)$ is unbounded at $x=0$ if $g(0)\neq 0$. Moreover if $g(0)=0$, the derivative $f'(x)$ of the solution $f(x)$ of (1.1) still may be unbounded at the origin (see [2]). These properties bring on much difficulty for the numerical treatment of (1.1).

There are many classes of numerical methods that have been developed over past few decades for the approximate solution of Eq. (1.1), such as product-integration methods [1,7], collocation methods [4–6], backward Euler methods [8,10] and fractional multistep methods [22], etc. Unfortunately, the first kind Abel integral equation (1.1) is often regarded as an ill-posed problem. The numerical treatment is more difficult for first kind Abel integral equations than for second kind ones, which have been widely studied (see, e.g., [1,2,15,16,21]). For (1.2), Gorenflo [9,10] presented some numerical methods of the fractional calculus, e.g., the Grünwald–Letnikov difference approximation

$$D_h^{1-\alpha}f(x)=h^{\alpha-1}(\delta_h^{1-\alpha}g)(x)=h^{\alpha-1}\sum_{j=0}^{[x/h+(1-\alpha)/2]}(-1)^j\binom{1-\alpha}{j}g(x+((1-\alpha)/2-j)h). \tag{1.4}$$

The formula (1.4) has accuracy order $\mathrm{O}(h^2)$ if $g(x)$ is sufficiently smooth and vanish at $x\leqslant 0$, else has accuracy order $\mathrm{O}(h)$.

On the other hand, Lubich [14] introduced a fractional multistep method for Abel–Volterra integral equation of the first kind, and Plato [22] considered fractional multistep methods for weakly singular Volterra integral equations of the first kind with perturbed right-hand side. In this paper, we propose a completely different approach for solving (1.1).

Unlike the first kind Fredholm integral equation, which essentially is an ill-posed problem, the first kind Abel integral equation can be transformed into the second kind Abel integral equation by a standard method (see [1]). If we replace x by s in (1.1), multiply both sides by $(x-s)_+^{\alpha-1}$, and then integrate with respect to s, (1.1) can be written as

$$\int_0^x L(x,y)f(y)\,\mathrm{d}y=G(x), \tag{1.5}$$

where

$$L(x,y)=\int_y^x\frac{H(s,y)}{(x-s)^{1-\alpha}(s-y)^\alpha}\,\mathrm{d}s=\int_0^1\frac{H(y+\tau(x-y),y)}{(1-\tau)^{1-\alpha}\tau^\alpha}\,\mathrm{d}\tau, \tag{1.6}$$

and

$$G(x)=\int_0^x\frac{g(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y=x^\alpha\int_0^1\frac{g(x\tau)}{(1-\tau)^{1-\alpha}}\,\mathrm{d}\tau. \tag{1.7}$$

Since $L(x,x)=H(x,x)\pi/\sin(\pi\alpha)\neq 0$ for $0\leqslant x\leqslant 1$ and $G(0)=0$, differentiating (1.5) with respect to x, we get

$$f(x)+\int_0^x\tilde{L}(x,y)f(y)\,\mathrm{d}y=v(x)\quad(0\leqslant x\leqslant 1), \tag{1.8}$$

where $\tilde{L}(x,y)=L_x(x,y)/L(x,x)$ and $v(x)=G'(x)/L(x,x)$. Thus, (1.1) is transformed into a second kind Volterra integral equation, whose kernel and the right-hand side term are expressed by weakly singular integrals.

Since the solution $f(x)$ of (1.1) or its derivative $f'(x)$ may be unbounded at the origin, Baratella and Orsi [2] proposed to take the change of variable $x=\gamma(t)=t^q$ in (1.8), where q is an undetermined positive constant. Then (1.8) is written as

$$f(\gamma(t))+\int_0^{\gamma(t)}\tilde{L}(\gamma(t),y)f(y)\,\mathrm{d}y=v(\gamma(t))\quad(0\leqslant t\leqslant 1).$$

Letting $y=\gamma(s)$, we have

$$f(\gamma(t))+\int_0^t \tilde{L}(\gamma(t),\gamma(s))f(\gamma(s))\gamma'(s)\,\mathrm{d}s=v(\gamma(t))\quad (0\leqslant t\leqslant 1). \tag{1.9}$$

Multiply (1.9) by $\gamma'(t)$ and set

$$u(t)=\gamma'(t)f(\gamma(t)),\quad \eta(t)=\gamma'(t)v(\gamma(t)),\quad \bar{L}(t,s)=\gamma'(t)\tilde{L}(\gamma(t),\gamma(s)),$$

then Eq. (1.9) is simplified as

$$u(t)+\int_0^t \bar{L}(t,s)u(s)\,\mathrm{d}s=\eta(t)\quad (0\leqslant t\leqslant 1). \tag{1.10}$$

We shall prove that a suitable choice of the parameter q can ensure that the solution $u(t)$ and $\eta(t)$ of (1.10) are sufficiently smooth (see Remark 2).

In this paper, we focus on mechanical quadrature methods and their extrapolations for solving (1.10). It is well known that Richardson extrapolation as an accelerating convergence technique has been applied to many fields in computing mathematics (see [3,13,18]), e.g., the extrapolation has been applied to solve Volterra integral equations of the second kind with continuous kernel (see [1,18]). However, to the best of our knowledge, it is the first attempt to apply Richardson extrapolation to the first kind Abel integral equation. In Section 2 two algorithms are constructed based on the trapezoidal and the mid-point quadrature rules, and in Section 3 the convergence and the error estimate are proved. Then in Section 4 we prove that algorithms possess high accuracy and asymptotic expansions of the errors, that is, Richardson extrapolation can be used to obtain a higher accuracy order and an a posteriori error estimate is obtained. Some numerical results and comparisons show the efficiency of our methods in the last section.

Remark 1. Using the operator notation for fractional integration and differentiation and denoting by the operator of ordinary differentiation and applying the rules for their combinations, as e.g. described in [9–11], we observe, see (1.7) and (1.8), that

$$G'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{g(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y=\Gamma(\alpha)DJ^{\alpha}g=\Gamma(\alpha)D^{1-\alpha}g$$

is the solution of an Abel integral equation

$$J^{1-\alpha}G'=\Gamma(\alpha)g(x). \tag{1.11}$$

So, discretized fractional calculus, e.g., Grünwald–Letnikov difference $h^{\alpha-1}(\delta_h^{1-\alpha}g)(x)$ can be used to approximate $G'(x)$. Since $g(x)$ is possibly contaminated by noise, $g(x)$ in (1.11) will be replaced by $\hat{g}^{\varepsilon}=g(x)+g^{\varepsilon}(x)$, where $|g^{\varepsilon}(x)|\leqslant\varepsilon$. Gorenflo [9,10] presented error estimate of the perturbed approximation $\hat{G}^{\varepsilon}(x)$.

Remark 2. Assume that $g(x)\in C^k[0,1]$, $g(0)\neq 0$, then $f(x)=\mathrm{O}(x^{-\alpha})$ and $G'(x)=\mathrm{O}(x^{\alpha-1})$ as $x\to 0$. Therefore, $u(t)=\gamma'(t)f(\gamma(t))=\mathrm{O}(t^{q-1-q\alpha})$ and $\eta(t)=\gamma'(t)v(\gamma(t))=\mathrm{O}(t^{q-1+(\alpha-1)q})$, as $t\to 0$. We can choose $q\geqslant\max\{(k+1)/\alpha,(k+1)/(1-\alpha)\}$, such that the solution $u(t)$ and the right-hand side $\eta(t)$ of the transformed equation (1.10) belong to $C^k[0,1]$.

2. The numerical methods

Since the kernel and the right-hand side function of the integral equation (1.10) are expressed by weakly singular integrals, numerical methods must be considered for computing integrals with weak singularity at the end points. For this purpose, Navot's quadrature rule is used.

Lemma 2.1 (*Lyness and Ninham [17]; Navot [20]*). *Let* $g(x)\in C^{2m+1}[a,b](m\in Z^+)$, $G(x)=(b-x)^{\alpha}g(x)$, $h=(b-a)/N$ *and* $x_k=a+kh$ *for* $k=0,\ldots,N$, *then the modified trapezoidal integration rule* $T_N(g)$

has an asymptotic expansion

$$\begin{aligned} T_N(G) &= \frac{h}{2}G(x_0) + h\sum_{k=1}^{N-1} G(x_k) - \zeta(-\alpha)g(b)h^{1+\alpha} \\ &= \int_a^b (b-x)^\alpha g(x)\,\mathrm{d}x + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} G^{(2j-1)}(a)h^{2j} \\ &\quad + \sum_{j=1}^{2m} (-1)^j \zeta(-\alpha-j)\frac{g^{(j)}(b)h^{j+\alpha+1}}{j!} + \mathrm{O}(h^{2m+1}), \end{aligned} \tag{2.1}$$

where $\alpha > -1$, $\zeta(x)$ *is the Riemann–Zeta function and* B_{2j} $(j = 1, \ldots, m)$ *are the Bernoulli numbers with* $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$.

It is well known that periodization methods play significant roles in increasing accuracy of quadrature rules. In this paper, we will use a special periodization method called Sidi's $\sin^m$-transformation [23], which is constructed by

$$\psi_m(y) = \frac{\theta_m(y)}{\theta_m(1)}, \tag{2.2}$$

with

$$\theta_m(y) = \int_0^y \sin(\pi t)^m\,\mathrm{d}t, \quad m = 1, 2, \ldots .$$

Sidi derived

$$\begin{aligned} \psi_1(y) &= \tfrac{1}{2}(1-\cos(\pi y)), \\ \psi_2(y) &= \frac{1}{2\pi}(2\pi y - \sin(2\pi y)), \\ \psi_3(y) &= \tfrac{1}{16}(8 - 9\cos(\pi y) + \cos(3\pi y)), \\ \psi_4(y) &= \frac{1}{12\pi}(12\pi y - 8\sin(2\pi y) + \sin(4\pi y)). \end{aligned}$$

Obviously $\psi_m : [0, 1] \to [0, 1]$ is one-to-one and satisfies

$$\psi_m^{(2i-1)}(0) = \psi_m^{(2i-1)}(1), \quad i = 1, 2, \ldots .$$

$\psi_m(y)$ has asymptotic expansions

$$\begin{cases} \psi_m(y) \sim \varepsilon_m y^{m+1} + \sum\limits_{i=1}^{\infty} \varepsilon_{m,i}\, y^{m+1+2i} & \text{as } y \to 0+, \\ \psi_m(y) \sim 1 - \hat{\varepsilon}_m (1-y)^{m+1} - \sum\limits_{i=1}^{\infty} \hat{\varepsilon}_{m,i} (1-y)^{m+1+2i} & \text{as } y \to 1-, \end{cases} \tag{2.3}$$

where $\varepsilon_m \neq 0$, $\hat{\varepsilon}_m \neq 0$, $\varepsilon_{m,i}$, $\hat{\varepsilon}_{m,i}$ $(i = 1, 2, \ldots)$ are all constants.

In (1.5),

$$L_x(x, y) = \frac{\partial}{\partial x} L(x, y) = \int_0^1 H_1'(y + \tau(x-y), y)\frac{\tau^{1-\alpha}}{(1-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \tag{2.4}$$

where $H_1'(x,y)=(\partial/\partial x)H(x,y)$. Letting $\tau=\psi_m(t)$, then

$$L_x(x,y)=\int_0^1 \Phi(x,y;t)\,\mathrm{d}t, \tag{2.5}$$

with

$$\Phi(x,y;t)=H_1'(y+\psi_m(t)(x-y),y)\,\frac{\psi_m(t)^{1-\alpha}}{(1-\psi_m(t))^{1-\alpha}}\psi_m'(t). \tag{2.6}$$

By (2.3), there are functions $B(t)$ and $C(t)$, such that

$$\frac{\psi_m'(t)}{(1-\psi_m(t))^{1-\alpha}}=(m+1)\hat{\varepsilon}_m^{\alpha}(1-t)^{(m+1)\alpha-1}\,\frac{1-C(t)(1-t)^2}{(1-B(t)(1-t)^2/\hat{\varepsilon}_m^{\alpha})^{1-\alpha}},$$

which has a zero or pole of βth order at $t=1$, where $\beta=(m+1)\alpha-1$. Therefore, we can denote

$$\Phi(x,y;t)=(1-t)^{\beta}\phi(x,y;t),$$

where

$$\phi(x,y;t)=(m+1)\hat{\varepsilon}_m^{\alpha}H_1'(y+\psi_m(t)(x-y),y)\,\frac{\psi_m(t)^{1-\alpha}(1-C(t)(1-t)^2)}{\left(1-B(t)(1-t)^2/\hat{\varepsilon}_m^{\alpha}\right)^{1-\alpha}}, \tag{2.7}$$

and ϕ is nonsingular at $t=1$.

From Lemma 2.1 we may derive an approximation $L_x^h(x,y)$ of $L_x(x,y)$ as follows:

$$L_x^h(x,y)=\sum_{j=1}^{N-1} h\Phi(x,y;t_j)-(m+1)h^{\beta+1}\zeta(-\beta)H_1'(x,y)\hat{\varepsilon}_m^{\alpha}\quad (0\leqslant x\leqslant y\leqslant 1), \tag{2.8}$$

where $\hat{\varepsilon}_m$ is the constant in (2.3) and $t_j=jh,\ j=0,1,2,\ldots,N,\ h=1/N$.

By (1.7),

$$\begin{aligned}G'(x)&=\alpha x^{\alpha-1}\int_0^1\frac{g(x\tau)}{(1-\tau)^{1-\alpha}}\,\mathrm{d}\tau+x^{\alpha}\int_0^1\frac{g'(x\tau)\tau}{(1-\tau)^{1-\alpha}}\,\mathrm{d}\tau\\&=\alpha x^{\alpha-1}\int_0^1 G_1(x,\tau)\,\mathrm{d}\tau+x^{\alpha}\int_0^1 G_2(x,\tau)\,\mathrm{d}\tau,\end{aligned} \tag{2.9}$$

where

$$G_1(x;t)=\frac{g(x\psi_m(t))}{(1-\psi_m(t))^{1-\alpha}}\,\psi_m'(t)\quad\text{and}\quad G_2(x;t)=\frac{g'(x\psi_m(t))}{(1-\psi_m(t))^{1-\alpha}}\,\psi_m(t)\psi_m'(t). \tag{2.10}$$

In (1.10),

$$\begin{aligned}\eta(t)&=\frac{\gamma'(t)G'(\gamma(t))}{L(\gamma(t),\gamma(t))}=\frac{\alpha\gamma'(t)\gamma(t)^{\alpha-1}}{L(\gamma(t),\gamma(t))}\int_0^1 G_1(\gamma(t),\tau)\,\mathrm{d}\tau+\frac{\gamma'(t)\gamma(t)^{\alpha}}{L(\gamma(t),\gamma(t))}\int_0^1 G_2(\gamma(t),\tau)\,\mathrm{d}\tau\\&=I_1(t)+I_2(t).\end{aligned}$$

Since $G'(x)=\mathrm{O}(x^{\alpha-1})$, as $x\to 0$, we have that $I_1(t)=\mathrm{O}(t^{q\alpha-1})$ and $I_2(t)=\mathrm{O}(t^{q\alpha+q-1})$, as $t\to 0$. Choosing $q>1/\alpha$, we get $I_1(0)=I_2(0)=0$. It is similar to the derivation of (2.8) that we can obtain approximate expressions of $I_1(t)$ and $I_2(t)$ as

$$I_1^h(t)=\frac{\alpha\gamma'(t)\gamma(t)^{\alpha-1}}{L(\gamma(t),\gamma(t))}\left[\sum_{j=1}^{N-1} hG_1(\gamma(t),t_j)-(m+1)h^{\beta+1}\zeta(-\beta)g(\gamma(t))\hat{\varepsilon}_m^{\alpha}\right]\quad (0<t\leqslant 1), \tag{2.11}$$

and

$$I_2^h(t)=\frac{\gamma'(t)\gamma(t)^{\alpha}}{L(\gamma(t),\gamma(t))}\left[\sum_{j=1}^{N-1}hG_2(\gamma(t),t_j)-(m+1)h^{\beta+1}\zeta(-\beta)g'(\gamma(t))\hat{\varepsilon}_m^{\alpha}\right]\quad(0<t\leqslant 1). \tag{2.12}$$

Then

$$\eta^h(t)=I_1^h(t)+I_2^h(t) \tag{2.13}$$

is an approximation of $\eta(t)$ in (1.8). For the kernel $\bar{L}(x,y)$ of (1.10), the corresponding approximate expression is

$$\bar{L}^h(t,s)=\gamma'(t)\,\frac{L_x^h(\gamma(t),\gamma(s))}{L(\gamma(t),\gamma(t))}, \tag{2.14}$$

where $L(x,x)=H(x,x)\pi/\sin(\pi\alpha)\neq 0$.

By (2.13) and (2.14), an approximate integral equation

$$u(t)+\int_0^t \bar{L}^h(t,s)u(s)\,\mathrm{d}s=\eta^h(t) \tag{2.15}$$

of (1.10) is derived. Eq. (2.15) has to be discretized. Next we use the mid-point and trapezoidal rules to derive the numerical solution of (2.15).

Algorithm 1 (*The mid-point rectangular quadrature method*).

$$\begin{cases} u_0^M=\eta^h(t_0)=0, \\ u_i^M=\eta^h(t_i)-h\sum_{j=0}^{i-1}\bar{L}^h(t_i,t_{j+1/2})\,\dfrac{u_j^M+u_{j+1}^M}{2}, \quad i=1,2,\ldots,N, \end{cases} \tag{2.16}$$

where $t_i=ih$, $t_{j+1/2}=(j+1/2)h$ and $h=1/N$.

Algorithm 2 (*The trapezoidal quadrature method*).

$$\begin{cases} u_0^T=\eta^h(t_0)=0, \\ u_i^T=\eta^h(t_i)-h\sum_{j=0}^{i}\omega_{ij}\bar{L}^h(t_i,t_j)u_j^T, \quad i=1,2,\ldots,N, \end{cases} \tag{2.17}$$

where $\omega_{i0}=\omega_{ii}=\frac{1}{2},\ \omega_{ij}=1, 0<j<i, i=0,1,\ldots,N$.

As soon as $\{u_i^M\}$ and $\{u_i^T\}$ are found, $\{u_i^M/\gamma'(t_i)\}$ and $\{u_i^T/\gamma'(t_i)\}$ will be the approximations of the solution $\{f(\gamma(t_i))\}$ of (1.1).

3. Convergence and error estimate

Assume that the data function is not perturbed by noise, then approximations $\bar{L}^h(t,s)$ and $\eta^h(t)$ satisfy the following Lemma.

Lemma 3.1. *Let $H(x,\cdot), g(x)\in C^4[0,1]$, then the errors $\bar{L}^h(t,s)-\bar{L}(t,s)$ and $\eta^h(t)-\eta(t)$ have the estimates*

$$\bar{L}^h(t,s)-\bar{L}(t,s)=\mathrm{O}(h^{\lambda}), \tag{3.1}$$

and

$$\eta^h(t)-\eta(t)=\mathrm{O}(h^{\lambda}), \tag{3.2}$$

where $\lambda=\min\{3,\beta+3\}$.

Proof. Obviously, from $H(x, \cdot), g(x) \in C^4[0, 1]$, it follows that $\phi(\cdot, \cdot; t)$, $G_1(\cdot; t)(1-t)^{-\beta}$, $G_2(\cdot; t)(1-t)^{-\beta} \in C^3[0, 1]$. By Lemma 2.1, we have

$$\tilde{L}^h(x, y) - \tilde{L}(x, y) = T_1(x, y)h^{\beta+2} + T_2(x, y)h^2 + \mathrm{O}(h^\lambda), \tag{3.3}$$

with

$$T_1(x, y) = -\zeta(-\beta - 1)\frac{\partial}{\partial t}\left.\frac{\phi(x, y; t)}{L(x, x)}\right|_{t=1}, \quad T_2(x, y) = \frac{B_2}{2}\frac{\partial}{\partial t}\left.\frac{\phi(x, y; t)}{L(x, x)}\right|_{t=0}.$$

In (2.7), letting

$$R(t) = \frac{1 - C(t)(1-t)^2}{(1 - B(t)(1-t)^2/\hat{\varepsilon}_m^\alpha)^{1-\alpha}},$$

then

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y; t) = (m+1)\hat{\varepsilon}_m^\alpha(&H_1''(y + \psi_m(t)(x-y), y)(x-y)\psi_m'(t)R(t)\psi_m(t)^{1-\alpha} \\ &+ H_1'(y + \psi_m(t)(x-y), y)\psi_m(t)^{1-\alpha}R'(t) \\ &+ H_1'(y + \psi_m(t)(x-y), y)R(t)\psi_m(t)^{-\alpha}\psi_m'(t)).\end{aligned}$$

By $\psi_m'(1) = 0$, $R'(1) = 0$, we get $(\partial/\partial t)\phi(x, y; 1) = 0$, which implies that $T_1(x, y) = 0$. Similarly from $\psi_m(0) = 0$, $\psi_m'(0) = 0$, we have $(\partial/\partial t)\phi(x, y; 0) = 0$, then $T_2(x, y) = 0$. By $\bar{L}(t, s) = \gamma'(t)\tilde{L}(\gamma(t), \gamma(s))$, (3.1) is proved. Using the same technique we can also prove (3.2). This completes the proof of Lemma 3.1. □

In order to obtain an error estimate of our algorithms, we need the following discrete Gronwall inequality.

Lemma 3.2 (*Marchuk and Shaidurov [18]; McKee [19]*). *If a nonnegative sequence* $\{y_n,\ n = 0, \ldots, N\}$ *satisfies*

$$y_0 = 0, \quad y_n \leqslant A + Bh\sum_{j=0}^{n-1} y_j, \quad 1 \leqslant n \leqslant N, \quad h = 1/N,$$

then

$$\max_{0 \leqslant i \leqslant N} y_i \leqslant Ae^B,$$

where A and B are positive constants independent of h.

The errors of Algorithm 1 are estimated as follows.

Theorem 3.1. *Assume that* $H(x, \cdot), g(x) \in C^4[0, 1]$, $H(\cdot, y) \in C^3[0, 1]$ *and the step width h is sufficiently small, then the errors* $e_i^M = u(t_i) - u_i^M$, $i = 0, 1, 2, \ldots, N$, *of Algorithm* 1 *are obtained by*

$$\max_{1 \leqslant i \leqslant N} |e_i^M| \leqslant C_M h^2, \tag{3.4}$$

where C_M *is a constant independent of h.*

Proof. By Euler–Maclaurin formula (see [13]) of the mid-point rectangular integration rule, (1.10) becomes the following equality:

$$\begin{cases}\eta(t_0) = u(t_0) = 0, \\ \eta(t_i) = u(t_i) + \int_0^{t_i} \bar{L}(t_i, s)u(s)\,\mathrm{d}s \\ \qquad = u(t_i) + h\sum_{j=0}^{i-1}\bar{L}(t_i, t_{j+1/2})u(t_{j+1/2}) + Q_1(t_i)h^2 + \mathrm{O}(h^3), \quad i = 1, 2, \ldots, N,\end{cases} \tag{3.5}$$

with

$$Q_1(t_i) = -\frac{C_2}{2}\frac{\mathrm{d}}{\mathrm{d}s}(\bar{L}(t_i, s)u(s))\bigg|_{s=0}^{s=t_i}, \quad C_2 = -(1-2^{-1})B_2.$$

By (3.2) and (3.5) we have

$$\begin{aligned}\eta^h(t_i) &= \eta(t_i) + \mathrm{O}(h^\lambda)\\ &= u(t_i) + h\sum_{j=0}^{i-1}\bar{L}(t_i, t_{j+1/2})u_{j+1/2} + Q_1(t_i)h^2 + \mathrm{O}(h^\lambda).\end{aligned}$$

Since the solution of (2.15) is sufficiently smooth, we may assume $u(t) \in C^3[0, 1]$. From Taylor's expansion, it follows that

$$\frac{u(t_j) + u(t_{j+1})}{2} = u(t_{j+1/2}) + u''(t_{j+1/2})\frac{h^2}{8} + \mathrm{O}(h^3).$$

So

$$\begin{aligned}\eta^h(t_i) &= u(t_i) + h\sum_{j=0}^{i-1}\bar{L}(t_i, t_{j+1/2})u(t_{j+1/2}) + Q_1(t_i)h^2 + \mathrm{O}(h^\lambda)\\ &= u(t_i) + h\sum_{j=0}^{i-1}\bar{L}(t_i, t_{j+1/2})\left(\frac{u(t_j) + u(t_{j+1})}{2} - u''(t_{j+1/2})\frac{h^2}{8} + \mathrm{O}(h^3)\right)\\ &\quad + Q_1(t_i)h^2 + \mathrm{O}(h^\lambda).\end{aligned} \tag{3.6}$$

For given $u(t)$, let

$$Q_2(t) = \frac{1}{8}\int_0^x \bar{L}(t, s)u''(s)\,\mathrm{d}s. \tag{3.7}$$

Obviously,

$$Q_2(t_i) = \frac{1}{8}\sum_{j=0}^{i-1} h\bar{L}(t_i, t_{j+1/2})u''(t_{j+1/2}) + \mathrm{O}(h^2). \tag{3.8}$$

Substituting (3.8) into (3.6), we get

$$\begin{aligned}\eta^h(t_i) &= u(t_i) + h\sum_{j=0}^{i-1}\bar{L}(t_i, t_{j+1/2})\frac{u(t_j) + u(t_{j+1})}{2} + (Q_1(t_i) - Q_2(t_i))h^2 + \mathrm{O}(h^\lambda)\\ &= u(t_i) + h\sum_{j=0}^{i-1}(\bar{L}^h(t_i, t_{j+1/2}) - \mathrm{O}(h^\lambda))\frac{u(t_j) + u(t_{j+1})}{2} + (Q_1(t_i) - Q_2(t_i))h^2 + \mathrm{O}(h^\lambda)\\ &= u(t_i) + h\sum_{j=0}^{i-1}\bar{L}^h(t_i, t_{j+1/2})\frac{u(t_j) + u(t_{j+1})}{2} + (Q_1(t_i) - Q_2(t_i))h^2 + \mathrm{O}(h^\lambda),\end{aligned} \tag{3.9}$$

where $\lambda = \min\{3, \beta + 3\}$. Subtracting (3.9) from (2.16), we get

$$\begin{cases} e_0^M = 0,\\ e_i^M = -h\sum_{j=0}^{i-1}\bar{L}^h(t_i, t_{j+1/2})\dfrac{e_j + e_{j+1}}{2} + E_{i,t}(t_i, t, u(t)),\end{cases} \tag{3.10}$$

where

$$E_{i,t}(t_i, t, u(t)) = -(Q_1(t_i) - Q_2(t_i))h^2 + \mathrm{O}(h^\lambda).$$

Let

$$A = \max_{1 \leqslant i \leqslant N} \max_{0 \leqslant t \leqslant 1} |E_{i,t}(t_i, t, u(t))|,$$

and

$$B = \sup_{h>0} \max_{1 \leqslant i \leqslant N} \max_{0 \leqslant j \leqslant i-1} |\bar{L}^h(t_i, t_{j+1/2})|.$$

Since $-1 < \beta < m+1$, we can easily derive $A = \mathrm{O}(h^2)$. Let h be sufficiently small so that

$$|h\bar{L}^h(t_i, t_{i-1/2})| \leqslant 1,$$

then

$$|e_i| \leqslant A + 2Bh \sum_{j=1}^{i-1} |e_j|.$$

By Lemma 3.2, there is a constant C_M independent of h satisfying

$$\max_{1 \leqslant i \leqslant N} |e_i| \leqslant C_M h^2.$$

This completes the proof. $\square$

For Algorithm 2, we have similar estimate.

Theorem 3.2. *Assume that $H(x, \cdot), g(x) \in C^4[0, 1]$, $H(\cdot, y) \in C^3[0, 1]$, and the step width h is sufficiently small, then the errors $e_i^T = u(t_i) - u_i^T$, $i = 0, 1, \ldots, N$, of Algorithm 2 have the estimate*

$$\max_{1 \leqslant i \leqslant N} |e_i^T| \leqslant C_T h^2,$$

where C_T is a constant dependent of h.

4. Asymptotic expansion, extrapolation and a posteriori estimate

Now we shall give asymptotic expansions of the errors of Algorithms 1 and 2, which are foundations of extrapolation and a posteriori estimate.

For the errors of Algorithm 1, by (3.10) we have

$$\begin{cases} e_0^M = 0, \\ e_i^M = -h\sum_{j=0}^{i-1} \bar{L}^h(t_i, t_{j+1/2}) \dfrac{e_j^M + e_{j+1}^M}{2} - (Q_1(t_i) - Q_2(t_i))h^2 + \mathrm{O}(h^\lambda). \end{cases} \tag{4.1}$$

Now, we construct auxiliary problems. Let $\widehat{Q}_k(t)$ satisfy the following linear Volterra integral equations

$$\widehat{Q}_k(t) = Q_k(t) - \int_0^t \bar{L}(t, s)\widehat{Q}_k(s)\,\mathrm{d}s, \quad k = 1, 2,$$

and $\{\widehat{Q}_k^h(t_i)\}$ satisfy their approximate equations

$$\widehat{Q}_k^h(t_i) = Q_k(t_i) - h \sum_{j=0}^{i-1} \bar{L}(t_i, t_{j+1/2}) \frac{\widehat{Q}_k^h(t_j) + \widehat{Q}_k^h(t_{j+1})}{2}, \quad k = 1, 2, \ \ i = 0, 1, 2, \ldots, N. \tag{4.2}$$

It is similar to the proof of Theorem 3.1, we can obtain

$$\max_{0\leqslant i\leqslant N} |\widehat{Q}_k(t_i) - \widehat{Q}_k^h(t_i)| \leqslant \widehat{C}h^2, \quad k = 1, 2, \tag{4.3}$$

where $\widehat{C}$ is independent of h. Note that the approximate equation (4.2) can be rewritten as

$$\begin{aligned}\widehat{Q}_k^h(t_i) = &-h\sum_{j=0}^{i-1} \bar{L}^h(t_i, t_{j+1/2})\,\frac{\widehat{Q}_k^h(t_j) + \widehat{Q}_k^h(t_{j+1})}{2} + Q_k(t_i)\\ &-h\sum_{j=0}^{i-1} (\bar{L}(t_i, t_{j+1/2}) - \bar{L}^h(t_i, t_{j+1/2}))\,\frac{\widehat{Q}_k^h(t_j) + \widehat{Q}_k^h(t_{j+1})}{2}.\end{aligned} \tag{4.4}$$

Therefore, letting

$$\hat{e}_i^M = e_i^M - h^2(\widehat{Q}_1^h(t_i) - \widehat{Q}_2^h(t_i)),$$

putting (4.4) into (4.1), and from (3.1) we can derive

$$\begin{aligned}\hat{e}_i^M = &-h\sum_{j=0}^{i-1} \bar{L}^h(t_i, t_{j+1/2})\,\frac{e_j^M + e_{j+1}^M}{2} - (Q_1(t_i) - Q_2(t_i))h^2 + \mathrm{O}(h^\lambda)\\ &+h^3\sum_{j=0}^{i-1} \bar{L}^h(t_i, t_{j+1/2})\left(\frac{\widehat{Q}_1^h(t_j) + \widehat{Q}_1^h(t_{j+1})}{2} - \frac{\widehat{Q}_2^h(t_j) + \widehat{Q}_2^h(t_{j+1})}{2}\right) + (Q_1(t_i) - Q_2(t_i))h^2\\ &-h^3\sum_{j=0}^{i-1} (\bar{L}(t_i, t_{j+1/2}) - \bar{L}^h(t_i, t_{j+1/2}))\left(\frac{\widehat{Q}_1^h(t_j) + \widehat{Q}_1^h(t_{j+1})}{2} - \frac{\widehat{Q}_2^h(t_j) + \widehat{Q}_2^h(t_{j+1})}{2}\right),\end{aligned}$$

which can be simplified as

$$\hat{e}_i^M = -h\sum_{j=0}^{i-1} \bar{L}^h(t_i, t_{j+1/2})\,\frac{\hat{e}_j^M + \hat{e}_{j+1}^M}{2} + \mathrm{O}(h^\lambda).$$

From Lemma 3.2, there is a constant C satisfying

$$\max_{1\leqslant i\leqslant N} |\hat{e}_i^M| \leqslant Ch^\lambda. \tag{4.5}$$

Using (4.3), we prove that

$$u_i^M = u(t_i) - (\widehat{Q}_1(t_i) - \widehat{Q}_2(t_i))h^2 + \mathrm{O}(h^\lambda). \tag{4.6}$$

Similarly, we can prove that the asymptotic expansion of the errors for Algorithm 2 is

$$u_i^T = u(t_i) - \widehat{Q}_1^*(t_i)h^2 + \mathrm{O}(h^\lambda), \tag{4.7}$$

where

$$\widehat{Q}_1^*(t_i) = -2\widehat{Q}_1(t_i).$$

From above discussions, we have proved the following theorem.

Theorem 4.1. *If the solution $u(t) \in C^3[0, 1]$ in* (1.10), *and $\bar{L}(t, \cdot), \eta(t) \in C^3[0, 1]$, then there are functions $\widehat{Q}_1(t)$, $\widehat{Q}_2(t)$, $\widehat{Q}_1^*(t)$ independent of h, such that asymptotic expansions* (4.6) *and* (4.7) *hold.*

Expressions (4.6) and (4.7) imply that for Algorithms 1 and 2, Richardson h^2-extrapolation can be employed to obtain higher accuracy order. For example, using h^2-extrapolation for Algorithm 1, we have

$$u_i^{M^*}(h) = \frac{4u_i^M(h/2) - u_i^M(h)}{3}, \quad i = 1, 2, \ldots, N, \tag{4.8}$$

where $u_i^M(h)$ and $u_i^M(h/2)$ are computed by (2.16) according to h and $h/2$. From (4.6), the errors $u_i^M(h) - u(t_i)$ have accuracy $\mathrm{O}(h^\lambda)$. Furthermore, an a posteriori asymptotic error estimate

$$|u_i^M(h/2) - u(t_i)| \leqslant \left| \frac{u_i^M(h) - u_i^M(h/2)}{3} \right| + \mathrm{O}(h^\lambda) \tag{4.9}$$

is derived, that is, we can estimate the errors $|u_i^M(h/2) - u(t_i)|$ by $|(u_i^M(h) - u_i^M(h/2))/3|$.

The extrapolation and the a posteriori error estimate for Algorithm 2 are similarly used.

Remark 3. For given $0 < \alpha < 1$, we can choose m such that $\beta = (m+1)\alpha - 1 \geqslant 0$, then by means of Richardson extrapolation, an approximation u_L^M with a higher accuracy order $\mathrm{O}(h^3)$ can be obtained. Furthermore, if $\beta > 0$, and $\bar{L}(t, \cdot), \eta(t) \in C^5[0, 1]$, there are $\widetilde{Q}_1^T(t)$, $\widetilde{Q}_2^T(t)$, $\widetilde{Q}_1^M(t)$, $\widetilde{Q}_2^M(t)$ independent of h, such that

$$u_i^T = u(t_i) - \widetilde{Q}_1^T(t_i)h^2 - \widetilde{Q}_2^T(t_i)h^3 + \mathrm{O}(h^\mu),$$

and

$$u_i^M = u(t_i) - \widetilde{Q}_1^M(t_i)h^2 - \widetilde{Q}_2^M(t_i)h^3 + \mathrm{O}(h^\mu),$$

where $\mu = \min\{4, \beta+3\}$. It implies that continuing the h^3-extrapolation procedure, a higher accuracy order $\mathrm{O}(h^\mu)$ will be obtained.

5. Numerical examples

In this section, we apply Algorithms 1 and 2 and their extrapolation to several examples, where Sidi's $\sin^2$-transformation is involved, i.e., we use $m = 2$ in the equality (2.2), or, let $\tau = \psi_2(t) = (1/2\pi)(2\pi t - \sin(2\pi t))$ in (2.4). For the purpose of comparison, we also use product-integration method (PIM) in Baker [1], which results in the following discrete equations:

$$\begin{cases} f_0^P = \dfrac{G'(0)}{L(0,0)}; \\ g(x_i) = \int_0^x \dfrac{H(x_i, y)}{(x_i - y)^\alpha} f(y)\,\mathrm{d}y \approx \sum_{j=0}^{i-1} D_j(x_i) H(x_i, x_{j+1/2}) \dfrac{f_j^P + f_{j+1}^P}{2}, \quad i = 1, \ldots, N, \end{cases}$$

where

$$D_j(x_i) = \int_{x_j}^{x_{j+1}} (x_i - y)^{-\alpha}\,\mathrm{d}y = \frac{h^{1-\alpha}((i-j-1)^{1-\alpha} - (i-j)^{1-\alpha})}{\alpha - 1}, \quad j = 0, \ldots, i-1.$$

Example 1. Consider Eq. (1.1) with

$$\alpha = \frac{1}{4}, \quad H(x, y) = x^2y^3 + y^4 + 1, \quad g(x) = \frac{32768}{100947}x^{31/4} + \frac{262144}{908523}x^{27/4} + \frac{128}{231}x^{11/4},$$

which has exact solution $f(x) = x^2$. Then in (1.8), $\tilde{L}(x, y) = L_x(x, y)/L(x, x)$ and $v(x) = G'(x)/L(x, x)$, where

$$L(x, y) = \int_0^1 \frac{[y + \tau(x-y)]^2 y^3 + y^4 + 1}{(1-\tau)^{3/4}\tau^{1/4}}\,\mathrm{d}\tau,$$

Table 1
The errors for Example 1 at $x = 1.0$

h	Mid-poi.	a post.	h^2-extra.	Trapz.	a post.	h^2-extra.	PIM
0.05	1.7939e − 4			−4.9673e − 4			7.7199e − 3
0.025	4.4000e − 5	4.5130e − 5	−1.1293e − 6	−1.2343e − 4	1.2443e − 4	1.0072e − 6	2.4062e − 3
0.0125	1.0987e − 5	1.1004e − 5	−1.7174e − 8	−3.0843e − 5	3.0862e − 5	1.8779e − 8	7.4275e − 4

Table 2
The errors for Example 2 at $x = 1.0$

h	Mid-poi.	a post.	h^2-extra.	Trapz.	a post.	h^2-extra.	PIM
0.05	2.2008e − 3			−7.6311e − 3			−9.1054e − 1
0.025	5.6213e − 4	5.4622e − 4	1.5910e − 5	−1.8978e − 3	1.9111e − 3	1.3345e − 5	−6.4868e − 1
0.0125	1.4148e − 4	1.4022e − 4	1.2574e − 6	−4.7362e − 4	4.7473e − 4	1.0978e − 6	−4.6037e − 1

and

$$G(x) = x^{1/4}\int_0^1 \frac{1}{(1-\tau)^{3/4}}\left\{\frac{32768}{100947}(x\tau)^{31/4} + \frac{262144}{908523}(x\tau)^{27/4} + \frac{128}{231}(x\tau)^{11/4}\right\}\mathrm{d}\tau.$$

The errors of approximate solutions obtained by the mid-point and trapezoidal quadrature methods, their Richardson extrapolation and the product-integration method (PIM) at $x = 1.0$ are presented in Table 1.

Example 2. The integral equation (1.1) is considered with

$$\alpha = \tfrac{1}{2}, \quad H(x, y) = x^2 y + e, \quad g(x) = 3x^4\pi + 4e\pi x.$$

This equation has exact solution $f(x) = 8\sqrt{x}$. In (1.6), $\tilde{L}(x, y) = L_x(x, y)/L(x, x)$ and $v(x) = G'(x)/L(x, x)$, where

$$L(x, y) = \int_0^1 \frac{[y + \tau(x-y)]^2 y + e}{(1-\tau)^{1/2}\tau^{1/2}}\,\mathrm{d}\tau$$

and

$$G(x) = x^{1/2}\int_0^1 \frac{3(x\tau)^4\pi + 4e\pi x\tau}{(1-\tau)^{1/2}}\,\mathrm{d}\tau.$$

Using a smoothing transformation $x = \gamma(t) = t^2$, we obtain the integral equation (1.10) with $u(t) = \gamma'(t) f(\gamma(t)) = 2tf(t^2)$, $\eta(t) = 2tv(t^2)$, $\bar{L}(t, s) = 2t\tilde{L}(t^2, s^2)$. The errors are listed in Table 2.

Example 3. We consider Eq. (1.1) with

$$\alpha = \tfrac{1}{2}, \quad H(x, y) = \mathrm{e}^{-(x-y)/2}, \quad g(x) = 2\sqrt{x}\mathrm{e}^{-x/2} + (x-1)\sqrt{2\pi}\, erf\left(\frac{\sqrt{2x}}{2}\right),$$

where $erf(x)$ is the error function

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

Table 3
The errors for Example 3 at $x = 1.0$

h	Mid-poi.	a post.	h^2-extra.	Trapz.	a post.	h^2-extra.	PIM
0.05	−2.4117e − 5			3.5765e − 5			5.7044e − 3
0.025	−5.3423e − 6	6.2582e − 6	9.1604e − 7	9.6282e − 6	8.7123e − 6	9.1612e − 7	2.3481e − 3
0.0125	−1.2780e − 6	1.3548e − 6	7.6760e − 8	2.4646e − 6	2.3879e − 6	7.6759e − 8	9.9008e − 4

The exact solution is $f(x) = x$. The transformed equation (1.8) has $\tilde{L}(x, y) = L_x(x, y)/L(x, x)$ and $v(x) = G'(x)/L(x, x)$, where

$$L(x, y) = \int_0^1 \frac{1}{(1-\tau)^{1/2}\,\tau^{1/2}}\, \mathrm{e}^{-(x-y)\tau/2}\, \mathrm{d}\tau$$

and

$$G(x) = x^{1/2} \int_0^1 \frac{1}{(1-\tau)^{1/2}} \left\{ 2\sqrt{x\tau}\mathrm{e}^{-x\tau/2} + (x\tau - 1)\sqrt{2\pi}\, erf\left(\frac{\sqrt{2x\tau}}{2}\right) \right\} \mathrm{d}\tau.$$

The errors of algorithms are listed in Table 3.

Example 4. We consider the fractional integral equation

$$\frac{1}{\Gamma(1/2)} \int_0^x \frac{1}{(x-y)^{1/2}}\, f(y)\, \mathrm{d}y = \sqrt{\pi} \quad (0 \leqslant x \leqslant 1,\ 0 < \alpha < 1),$$

which is equivalent to the equation

$$\int_0^x \frac{x^2+1}{(x-y)^{1/2}}\, f(y)\, \mathrm{d}y = \pi x^2 + \pi \quad (0 \leqslant x \leqslant 1,\ 0 < \alpha < 1).$$

The exact solution is $f(x) = 1/\sqrt{x}$. Since the solution is unbounded at the origin, the product-integration method is not feasible. Using the same technique, we can transform it into a second kind Volterra integral equation as (1.8) with $\tilde{L}(x, y) = L_x(x, y)/L(x, x)$ and $v(x) = G'(x)/L(x, x)$, where

$$L(x, y) = \int_0^1 \frac{2[y + \tau(x-y)]}{(1-\tau)^{3/4}\tau^{1/4}}\, \mathrm{d}\tau,$$

and

$$G(x) = x^{1/4} \int_0^1 \frac{1}{(1-\tau)^{3/4}} \{(x\tau)^2\pi + \pi\}\, \mathrm{d}\tau.$$

By applying a smoothing transformation $x = \gamma(t) = t^4$, we obtain the integral equation (1.10) with $u(t) = \gamma'(t) f(\gamma(t)) = 4t^3 f(t^4) = 4t$, $\eta(t) = 4t^3 v(t^4)$, $\bar{L}(t, s) = 4t^3 \tilde{L}(t^4, s^4)$, in which the solution is bounded at the origin. By using the mid-point and trapezoidal quadrature methods, and Richardson extrapolation, we have the following errors table. Moreover errors obtained by the Grünwald–Letnikov difference approximation (GLDA) are also listed in Table 4.

The above results show that our numerical methods possess high accuracy, and at the same time, extrapolation procedures and the a posteriori estimate are very effective.

Remark 4. In general, h^3-extrapolation can increase further the accuracy. For example, in Example 4, continuing h^3-extrapolation procedure for mid-point rectangle rule and Table 4, we have the error table (Table 5).

Table 4
The errors for Example 4 at $x = 1.0$

h	Mid-poi.	a post.	h^2-extra.	Trapz.	a post.	h^2-extra.	GLDA
0.05	7.5283e − 4			−1.5341e − 3			−6.2299e − 3
0.025	1.8995e − 4	1.8763e − 4	2.3192e − 6	−3.8216e − 4	3.8398e − 4	1.8167e − 6	−3.1200e − 3
0.0125	4.7620e − 5	4.7443e − 5	1.7799e − 7	−9.5430e − 5	9.5577e − 5	1.4672e − 7	−1.5613e − 3

Table 5
The errors for Example 4 at $x = 1.0$

h	Mid-poi.	h^2-extra.	h^3-extra.
0.0125	4.7620e − 5	1.7799e − 7	−1.2790e − 7
0.00625	1.1916e − 5	1.4667e − 8	−8.6649e − 9

Acknowledgement

The authors would like to thank Professor Rudolf Gorenflo for his insightful suggestions.

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